What is a Path?
Tool: Configuration Space (C-Space C)
Configuration Space

q = (q₁, ..., qₙ)
Definition

A robot **configuration** is a specification of the positions of all robot points relative to a fixed coordinate system.

Usually a configuration is expressed as a "vector" of position/orientation parameters.
Rigid Robot Example

- 3-parameter representation: \( q = (x, y, \theta) \)
- In a 3-D workspace \( q \) would be of the form \( (x, y, z, \alpha, \beta, \gamma) \)
Articulated Robot Example

\[ q = (q_1, q_2, \ldots, q_{10}) \]
Protein example
Configuration Space of a Robot

- Space of all its possible configurations
- But the topology of this space is usually not that of a Cartesian space
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\[ C = S^1 \times S^1 \]
Configuration Space of a Robot

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\[ C = S^1 \times S^1 \]
What is its Topology?

$$(S1)^7 \times \mathbb{R}^3$$
Structure of Configuration Space

- It is a manifold
  For each point \( q \), there is a 1-to-1 map between a neighborhood of \( q \) and a Cartesian space \( \mathbb{R}^n \), where \( n \) is the dimension of \( C \)

- This map is a local coordinate system called a chart.
  \( C \) can always be covered by a finite number of charts. Such a set is called an atlas
Example
Case of a Planar Rigid Robot

- 3-parameter representation: $q = (x, y, \theta)$ with $\theta \in [0, 2\pi)$. Two charts are needed.
- Other representation: $q = (x, y, \cos \theta, \sin \theta)$
  → c-space is a 3-D cylinder $\mathbb{R}^2 \times S^1$
  embedded in a 4-D space
Rigid Robot in 3-D Workspace

- $q = (x, y, z, \alpha, \beta, \gamma)$

The c-space is a 6-D space (manifold) embedded in a 12-D Cartesian space. It is denoted by $R^3 \times SO(3)$

- Other representation: $q = (x, y, z, r_{11}, r_{12}, ..., r_{33})$ where $r_{11}$, $r_{12}$, ..., $r_{33}$ are the elements of rotation matrix $R$:

$$
\begin{pmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{pmatrix}
$$

with:

- $r_{i1}^2 + r_{i2}^2 + r_{i3}^2 = 1$
- $r_{i1}r_{j1} + r_{i2}r_{2j} + r_{i3}r_{3j} = 0$
- $\det(R) = +1$
Parameterization of SO(3)

- Euler angles: \((\phi, \theta, \psi)\)

- Unit quaternion: 
  \((\cos \theta/2, n_1 \sin \theta/2, n_2 \sin \theta/2, n_3 \sin \theta/2)\)
A metric or distance function $d$ in $C$ is a map

$$d: (q_1, q_2) \in C^2 \rightarrow d(q_1, q_2) \geq 0$$

such that:

- $d(q_1, q_2) = 0$ if and only if $q_1 = q_2$
- $d(q_1, q_2) = d(q_2, q_1)$
- $d(q_1, q_2) \leq d(q_1, q_3) + d(q_3, q_2)$
Metric in Configuration Space

Example:

- Robot A and point x of A
- \( x(q) \): location of x in the workspace when A is at configuration q
- A distance d in C is defined by:
  \[
  d(q,q') = \max_{x \in A} ||x(q)-x(q')||
  \]

where \( ||a - b|| \) denotes the Euclidean distance between points a and b in the workspace.
Specific Examples in $\mathbb{R}^2 \times S^1$

- $q = (x,y,\theta)$, $q' = (x',y',\theta')$ with $\theta, \theta' \in [0,2\pi)$
- $\alpha = \min\{|\theta-\theta'|, 2\pi-|\theta-\theta'|\}$

$d(q,q') = \sqrt{(x-x')^2 + (y-y')^2 + \alpha^2}$

$d(q,q') = \sqrt{(x-x')^2 + (y-y')^2 + (\alpha \rho)^2}$

where $\rho$ is the maximal distance between the reference point and a robot point
Notion of a Path

- A path in $C$ is a piece of continuous curve connecting two configurations $q$ and $q'$:
  \[ \tau : s \in [0,1] \rightarrow \tau(s) \in C \]
  \[ s' \rightarrow s \Rightarrow d(\tau(s),\tau(s')) \rightarrow 0 \]
Other Possible Constraints on Path

- Finite length, smoothness, curvature, etc...
- A trajectory is a path parameterized by time:
  \[ \tau : t \in [0, T] \rightarrow \tau(t) \in C \]
Obstacles in $C$-Space

- A configuration $q$ is **collision-free**, or **free**, if the robot placed at $q$ has null intersection with the obstacles in the workspace.
- The **free space** $F$ is the set of free configurations.
- A **$C$-obstacle** is the set of configurations where the robot collides with a given workspace obstacle.
- A configuration is **semi-free** if the robot at this configuration touches obstacles without overlap.
Disc Robot in 2-D Workspace
Rigid Robot Translating in 2-D

\[ CB = B \ominus A = \{b - a \mid a \in A, b \in B\} \]
Rigid Robot Translating in 2-D

\[ CB = B \ominus A = \{ b - a \mid a \in A, \ b \in B \} \]
Linear-Time Computation of C-Obstacle in 2-D

(convex polygons)

$O(n+m)$
Rigid Robot Translating and Rotating in 2-D
C-Obstacle for Articulated Robot
Free and Semi-Free Paths

- A free path lies entirely in the free space $F$
- A semi-free path lies entirely in the semi-free space
Remark on Free-Space Topology

• The robot and the obstacles are modeled as closed subsets, meaning that they contain their boundaries.

• One can show that the C-obstacles are closed subsets of the configuration space $C$ as well.

• Consequently, the free space $F$ is an open subset of $C$. Hence, each free configuration is the center of a ball of non-zero radius entirely contained in $F$.

• The semi-free space is a closed subset of $C$. Its boundary is a superset of the boundary of $F$. 
Notion of Homotopic Paths

- Two paths with the same endpoints are homotopic if one can be continuously deformed into the other.
- $\mathbb{R} \times S^1$ example:

  - $\tau_1$ and $\tau_2$ are homotopic.
  - $\tau_1$ and $\tau_3$ are not homotopic.
  - In this example, infinity of homotopy classes.
Connectedness of C-Space

- C is connected if every two configurations can be connected by a path.
- C is simply-connected if any two paths connecting the same endpoints are homotopic.
- Otherwise C is multiply-connected.

Examples:
- \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)
- \( S^1 \) and \( SO(3) \) are multiply-connected:
  - In \( S^1 \), infinity of homotopy classes
  - In \( SO(3) \), only two homotopy classes